Maxwell's Equations:

Apply Stokes theorem and divergence theorem

$$\nabla\cdot\vec{D}=\rho_v \longrightarrow \oint_s \vec{D}\cdot d\vec{s}=Q$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \longrightarrow \oint_c \vec{E}\cdot d\vec{l}=-\int_s \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s}$$

$$\nabla\cdot\vec{B}=0 \longrightarrow \oint_s \vec{B}\cdot d\vec{s}=0$$

$$\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \longrightarrow \oint_c \vec{H}\cdot d\vec{l}=\int_s\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)\cdot d\vec{s}$$

Electrostatics: $\left(\frac{\partial}{\partial t}\rightarrow 0\right)$

$\nabla\cdot\vec{D}=\rho_v$

$\nabla\times\vec{E}=0$

Magnetostatics:

$\nabla\cdot\vec{B}=0$

$\nabla\times\vec{H}=\vec{J}$

With no time variation, electric and magnetic fields are decoupled

Charge Density

Volume density $\rho_v \rightarrow Q=\int_v \rho_v\,dv$

surface density ρ_s

line charge density ρ_l

Current Density

$\vec{J}=\rho_v\vec{u}$ $\quad\vec{u}$ is velocity

$I=\int_s \vec{J}\cdot d\vec{s}$ total current flowing through surface s

Coulomb's Law

- Electric field due to a charge q: $\vec{E} = \hat{R}\dfrac{q}{4\pi\epsilon R^2}$
- Force on a charge q' due to $\vec{E}$: $\vec{F} = q'\vec{E}$

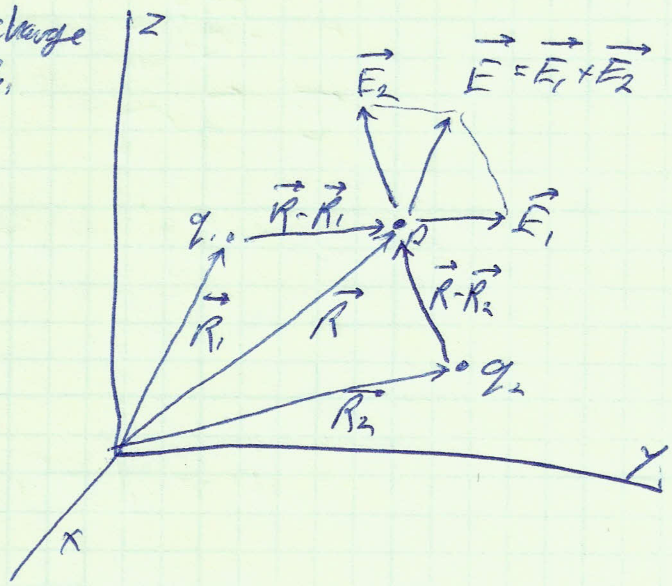
electric flux density $\vec{D} = \epsilon\vec{E} \leftarrow$ electric field intensity

$\epsilon = \epsilon_r\epsilon_0$, $\epsilon_0 = 8.854\times 10^{-12}$ F/m

- Electric field at point P (at vector $\vec{R}$ from origin) due to charges at vectors $R_{1\ldots N}$, with values $q_{1\ldots N}$

$$\vec{E}_1 = \frac{q_1(\vec{R}-\vec{R}_1)}{4\pi\epsilon|\vec{R}-\vec{R}_1|^3}$$ for charge q_1

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon}\sum_{i=1}^{N}\frac{q_i(\vec{R}-\vec{R}_i)}{|R-R_i|^3}}$$



- Electric field due to a charge distribution

$$\vec{E} = \frac{1}{4\pi\epsilon}\int_V \hat{R}\frac{\rho_v\,dV}{R^2}$$ volume distribution

$$\vec{E} = \frac{1}{4\pi\epsilon}\int_S \hat{R}\frac{\rho_s\,ds}{R^2}$$ surface distribution

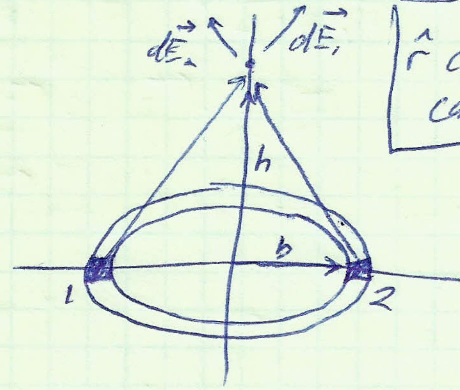
$$E = \frac{1}{4\pi\epsilon}\int_l \hat{R}\frac{\rho_l\,dl}{R^2}$$ line distribution

Example: Field from a ring, disk, or sheet of charge

ring of radius b

charge density ρ_ℓ

find $\vec{E}$ at $P(0,0,h)$



$\boxed{\hat{r} \text{ components cancel}}$

$$\vec{R}_1 = -\hat{r}b + \hat{z}h$$

$$R_1 = \sqrt{b^2+h^2}$$

$$\hat{R}_1 = \frac{-\hat{r}b + \hat{z}h}{\sqrt{b^2+h^2}}$$

$$\vec{E}_1 = \frac{q_1(\vec{R}_1)}{4\pi\epsilon|\vec{R}_1|^3} \quad \text{or} \quad d\vec{E} = \frac{1}{4\pi\epsilon}\frac{\hat{R}_1 \rho_\ell d\ell}{R_1^2}$$

$$dE_1 = \frac{1}{4\pi\epsilon_0}\hat{R}_1\frac{\rho_\ell d\ell}{R_1^2} = \frac{\rho_\ell b\, d\phi}{4\pi\epsilon_0}\frac{(-\hat{r}b + \hat{z}h)}{(b^2+h^2)^{3/2}}$$

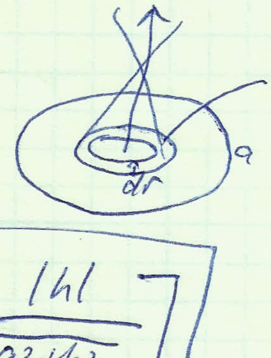
$$dE = \hat{z}\frac{\rho_\ell b h}{2\pi\epsilon_0} \cdot \frac{d\phi}{(b^2+h^2)^{3/3}} \longleftarrow \text{only integrate from 0 to } \pi$$

↑ factor of 2 because using both sides

$$\vec{E} = \hat{z}\frac{\rho_\ell b h}{2\pi\epsilon_0(b^2+h^2)^{3/2}}\int_0^\pi d\phi = \boxed{\hat{z}\frac{\rho_\ell b h}{2\epsilon_0(b^2+h^2)^{3/2}}}$$

For a disk of radius a:

replace $\rho_\ell b$ with $\rho_s r\, dr$



$$d\vec{E} = \hat{z}\frac{h\rho_s r\, dr}{2\epsilon_0(r^2+h^2)^{3/2}}$$

$$\vec{E} = \hat{z}\frac{\rho_s h}{2\epsilon_0}\int_0^a \frac{r\, dr}{(r^2+h^2)^{3/2}} = \boxed{\pm\hat{z}\frac{\rho_s}{2\epsilon_0}\left[1 - \frac{|h|}{\sqrt{a^2+h^2}}\right]}$$

switch variables to $x = r^2+h^2$ to solve integral

use physical understanding to get $\pm$ and $|h|$

If $a = \infty$ (infinite sheet)

then $\boxed{\vec{E} = \pm\hat{z}\frac{\rho_s}{2\epsilon_0}}$ independent of distance from sheet

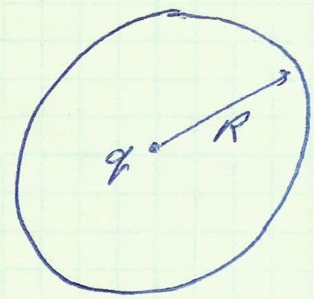
Gauss's Law

$$\nabla \cdot \vec{D} = \rho_v \longrightarrow \oint_s \vec{D} \cdot d\vec{s} = Q$$

Electric flux passing through a surface is equal to charge enclosed by that surface

- Assume spherical surface of radius R surrounding charge q

$\vec{D} = \hat{R} D_R$



$$\oint_s \vec{D} \cdot d\vec{s} = \oint_s \hat{R} D_R \cdot \hat{R} ds$$

$$= \oint_s D_R ds = D_R (4\pi R^2) = q$$

$$\vec{E} = \frac{\vec{D}}{\epsilon} = \hat{R} \frac{q}{4\pi\epsilon R^2} \rightarrow \text{Coulomb's law}$$

Example: Electric field of an infinite line charge

$$\int_s \vec{D} \cdot d\vec{s} = Q$$

$$\int_0^h \int_0^{2\pi} \hat{r} D_r \cdot \hat{r}\, r\, d\phi\, dz = \rho_\ell h$$

$$2\pi h D_r r = \rho_\ell h$$

$$\boxed{\vec{E} = \frac{\vec{D}}{\epsilon_0} = \hat{r} \frac{\rho_\ell}{2\pi\epsilon_0 r}}$$